EMBEDDING COPRODUCTS OF PARTITION LATTICES

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Dedicated to Béla Csákány for his 75th birthday

Abstract. We prove that the lattice $\mathrm{Eq}\,\Omega$ of all equivalence relations on an infinite set Ω contains, as a $0,1$-sublattice, the 0-coproduct of two copies of itself, thus answering a question by G. M. Bergman. Hence, by using methods initiated by de Bruijn and further developed by Bergman, we obtain that $\mathrm{Eq}\,\Omega$ also contains, as a sublattice, the coproduct of $2^{\mathrm{card}\,\Omega}$ copies of itself.

1. Introduction

Whitman's Theorem [10] states that every lattice L can be embedded into the lattice $\mathrm{Eq}\,\Omega$ of all equivalence relations on some set Ω. The cardinality of Ω may be taken equal to $\mathrm{card}\,L+\aleph_0$. There is not much room for improvement of the cardinality bound, as for example, $\mathrm{Eq}\,\Omega$ cannot be embedded into its dual lattice. (We believe the first printed occurrence of this result to be Proposition 6.2 in G. M. Bergman's recent preprint [2], although it may have already been known for some time.) Hence the question of embeddability into $\mathrm{Eq}\,\Omega$ of lattices of large cardinality (typically, $\mathrm{card}(\mathrm{Eq}\,\Omega)=2^{\mathrm{card}\,\Omega}$) is nontrivial.

In [2], Bergman also extends results of N. G. de Bruijn [3, 4] by proving various embedding results of large powers or copowers of structures such as symmetric groups, endomorphism rings, and monoids of self-maps of an infinite set Ω, into those same structures. The nature of the underlying general argument is categorical. The problem whether the lattice $\mathrm{Eq}\,\Omega$ contains a coproduct (sometimes called "free product" by universal algebraists) of two, or more, copies of itself, was stated as an open question in a preprint version of that paper. In the present note, we solve this problem in the affirmative.

The idea of our proof is the following. The lattice $\mathrm{Eq}\,\Omega$ of all equivalence relations on Ω is naturally isomorphic to the ideal lattice $\mathrm{Id}\,K$ of the lattice K of all *finitely generated* equivalence relations, that is, those equivalence relations containing only finitely many non-diagonal pairs. Denote by $K\amalg^0 K$ the coproduct (amalgamation) of two copies of K above the common ideal 0. As $K\amalg^0 K$ has the same cardinality as Ω, it follows from Jónsson's proof of Whitman's Embedding Theorem that the lattice $\mathrm{Id}(K\amalg^0 K)$ embeds into $\mathrm{Eq}\,\Omega$. Finally, we prove that the ideal lattice functor preserves the coproduct $\amalg^0$ and one-one-ness (Theorem 5.2), in such a way that $(\mathrm{Id}\,K)\amalg^0(\mathrm{Id}\,K)$ embeds into $\mathrm{Id}(K\amalg^0 K)$. Then it is easy to extend this result to the usual coproduct $(\mathrm{Id}\,K)\amalg(\mathrm{Id}\,K)$.

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We also present an example (Example 5.3) that shows that the result of Theorem 5.2 does not extend to amalgamation above a common (infinite) ideal. That is, for a common ideal A of lattices B and C, the canonical homomorphism from $(\operatorname{Id} B) \amalg_{\operatorname{Id} A} (\operatorname{Id} C)$ to $\operatorname{Id}(B \amalg_A C)$ may not be one-to-one.

2. Basic concepts

We refer to [7] for unexplained lattice-theoretical notions. For any subsets Q and X in a poset (i.e., partially ordered set) P, we put

$$Q \downarrow X = \{p \in Q \mid (\exists x \in X)(p \leq x)\} \quad \text{and} \quad Q \uparrow X = \{p \in Q \mid (\exists x \in X)(p \geq x)\}.$$

We also write $Q \downarrow x$, resp. $Q \uparrow x$ in case $X = \{x\}$. A subset Q of P is a *lower subset* of P if $Q = P \downarrow Q$.

A map $f\colon K \to L$ between lattices is *meet-complete* if for each $a \in K$ and each $X \subseteq K$, $a = \bigwedge X$ in K implies that $f(a) = \bigwedge f[X]$ in L. (Observe that we do not require either K or L to be a complete lattice.) When this is required only for nonempty X, we say that f is *nonempty-meet-complete*. *Join-completeness* and *nonempty-join-completeness* of maps are defined dually. We say that f is *complete* (resp., *nonempty-complete*) if it is both meet-complete and join-complete (resp., both nonempty-meet-complete and nonempty-join-complete). We say that f is *lower bounded* if $\{x \in K \mid y \leq f(x)\}$ is either empty or has a least element for each $y \in L$. *Upper bounded* homomorphisms are defined dually. Lower bounded homomorphisms are nonempty-meet-complete and upper bounded homomorphisms are nonempty-join-complete.

An *ideal* of a lattice L is a nonempty lower subset of L closed under finite joins. We denote by $\operatorname{Id} L$ the lattice of all ideals of L. For a lattice homomorphism $f\colon K \to L$, the map $\operatorname{Id} f\colon \operatorname{Id} K \to \operatorname{Id} L$ defined by

$$(\operatorname{Id} f)(X) = L \downarrow f[X]\,, \quad \text{for each } X \in \operatorname{Id} L\,,$$

is a nonempty-join-complete lattice embedding. If L is a 0-*lattice* (i.e., a lattice with least element), the canonical map $L \to \operatorname{Id} L$, $x \mapsto L{\downarrow}x$ is a 0-lattice embedding. The assignment that to every lattice associates its dual lattice L^{op} (i.e., the lattice with the same underlying set as L but reverse ordering) is a category equivalence—and even a category isomorphism—from the category of all lattices to itself, that sends 0-lattices to 1-lattices. For every lattice L, we denote by L° the lattice obtained by adding a new zero element to L.

A lattice L is *upper continuous* if for each $a \in L$ and each upward directed subset $\{x_i \mid i \in I\}$ of L admitting a join, the equality $a \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I}(a \wedge x_i)$ holds. We shall often use upper continuity in the following form: if I is an upward directed poset and both $(x_i \mid i \in I)$ and $(y_i \mid i \in I)$ are isotone families with respective joins x and y, then the family $(x_i \wedge y_i \mid i \in I)$ has join $x \wedge y$.

Every algebraic lattice is upper continuous, so, for example, $\operatorname{Id} L \cup \{\varnothing\}$ is upper continuous for any lattice L; hence $\operatorname{Id} L$ is also upper continuous. The lattice $\operatorname{Eq} \Omega$ of all equivalence relations on a set Ω, partially ordered by inclusion, is an algebraic lattice, thus it is upper continuous. Other examples of upper continuous lattices that are not necessarily complete are given in [1]. For example, it follows from [1, Corollary 2.2] that *every finitely presented lattice is upper continuous.*

We denote by $\mathfrak{P}(\Omega)$ the powerset of a set Ω, and by ω the set of all natural numbers.

3. The free lattice on a partial lattice

We recall Dean's description of the free lattice on a partial lattice, see [5] or [6, Section XI.9]. A *partial lattice* is a poset $(P, \leq)$ endowed with partial functions $\bigvee$ and $\bigwedge$ from the nonempty finite subsets of P to P such that if $p = \bigvee X$ (resp., $p = \bigwedge X$), then p is the greatest lower bound (resp., least upper bound) of X in P. An *o-ideal* of P is a lower subset A of P such that $p = \bigvee X$ and $X \subseteq A$ implies that $p \in A$ for each $p \in P$ and each nonempty finite subset X of P. The set $\overline{\mathrm{Id}}\, P$ of all o-ideals of P, partially ordered by inclusion, is an algebraic lattice. Observe that $\overline{\mathrm{Id}}\, P = (\mathrm{Id}\, P) \cup \{\varnothing\}$ in case P is a lattice. O-filters are defined dually; again, the lattice $\overline{\mathrm{Fil}}\, P$ of all o-filters of P, partially ordered by inclusion, is algebraic. We denote by $\mathcal{I}(A)$ (resp., $\mathcal{F}(A)$) the least o-ideal (resp., o-filter) of P containing a subset A of P.

The *free lattice* $\mathrm{F}_{\mathbf{L}}(P)$ on P is generated, as a lattice, by an isomorphic copy of P, that we shall identify with P. (The subscript $\mathbf{L}$ in $\mathrm{F}_{\mathbf{L}}(P)$ stands for the variety of all lattices, as the "free lattice on P" construction can be carried out in any variety of lattices.) For each $x \in \mathrm{F}_{\mathbf{L}}(P)$, the following subsets of P,

$$\mathcal{I}(x) = P \downarrow x = \{p \in P \mid p \leq x\} \quad \text{and} \quad \mathcal{F}(x) = P \uparrow x = \{p \in P \mid x \leq p\}$$

are, respectively, an o-ideal and an o-filter of P, which can also be evaluated by the following rules:

$$\mathcal{I}(x \vee y) = \mathcal{I}(x) \vee \mathcal{I}(y) \text{ in } \overline{\mathrm{Id}}\, P\,, \quad \mathcal{F}(x \vee y) = \mathcal{F}(x) \cap \mathcal{F}(y); \tag{3.1}$$

$$\mathcal{I}(x \wedge y) = \mathcal{I}(x) \cap \mathcal{I}(y), \quad \mathcal{F}(x \wedge y) = \mathcal{F}(x) \vee \mathcal{F}(y) \text{ in } \overline{\mathrm{Fil}}\, P\,, \tag{3.2}$$

for all $x, y \in \mathrm{F}_{\mathbf{L}}(P)$. The natural partial ordering on $\mathrm{F}_{\mathbf{L}}(P)$ satisfies the following "Whitman-type" condition:

$$\begin{aligned} x_0 \wedge x_1 \leq y_0 \vee y_1 \Longleftrightarrow &\text{ either } (\exists p \in P)(x_0 \wedge x_1 \leq p \leq y_0 \vee y_1) \\ &\text{or there is } i < 2 \text{ such that either } x_i \leq y_0 \vee y_1 \text{ or } x_0 \wedge x_1 \leq y_i\,, \end{aligned} \tag{3.3}$$

which is also the basis of the inductive definition of that ordering.

4. The 0-coproduct of a family of lattices with zero

Our development of the 0-coproduct of a family of lattices with zero below bears some similarities with the development of coproducts (called there *free products*) given in [7, Chapter VI]. Nevertheless, as we use the known results about the free lattice on a partial lattice (outlined in Section 3), our presentation becomes significantly shorter.

Let $(L_i \mid i \in I)$ be a family of lattices with zero. Modulo the harmless set-theoretical assumption that $L_i \cap L_j = \{0\}$ for all distinct indices $i, j \in I$, the *coproduct* (often called *free product* by universal algebraists) of $(L_i \mid i \in I)$ can be easily described as $\mathrm{F}_{\mathbf{L}}(P)$, where P is the partial lattice whose underlying set is the union $\bigcup_{i \in I} L_i$, whose underlying partial ordering is the one generated by the partial orders on all the L_is, and whose partial lattice structure consists of all existing joins and meets of nonempty finite subsets in each "component" L_i. We denote this lattice by $L = \coprod_{i \in I}^{0} L_i$, the superscript 0 meaning that the coproduct of the L_is is evaluated in the category of all 0-lattices and 0-preserving homomorphisms, which we shall often emphasize by saying "0-coproduct" instead of just coproduct. We shall also identify each L_i with its canonical copy in L. Of course, the coproduct

of any family of lattices $(L_i \mid i \in I)$ in the variety of all lattices is the sublattice of $\coprod_{i\in I}^{0} (L_i)^{\circ}$ generated by the union of the images of the L_is.

Now we shall analyze further the structure of the 0-coproduct L, in a fashion similar to the development in [7, Chapter VI]. We add a new largest element, denoted by ∞, to L, and we set $\overline{L}_i = L_i \cup \{\infty\}$ for each $i \in I$. The following lemma is an analogue, for 0-coproducts instead of coproducts, of [7, Theorem VI.1.10].

Lemma 4.1. *For each $x \in L$ and each $i \in I$, there are a largest element of L_i below x and a least element of $\overline{L}_i$ above x with respect to the ordering of $L \cup \{\infty\}$. Furthermore, if we denote these elements by $x_{(i)}$ and $x^{(i)}$, respectively, then the following formulas hold:*

$$\begin{aligned}
&p_{(i)} = p^{(i)} = p, \ \text{ if } p \in L_i;\\
&p_{(i)} = 0 \text{ and } p^{(i)} = \infty, \ \text{ if } p \in P \setminus L_i;\\
&(x \vee y)_{(i)} = x_{(i)} \vee y_{(i)} \text{ and } (x \wedge y)_{(i)} = x_{(i)} \wedge y_{(i)};\\
&(x \vee y)^{(i)} = x^{(i)} \vee y^{(i)};\\
&(x \wedge y)^{(i)} = \begin{cases} 0\,, & \text{if } x^{(j)} \wedge y^{(j)} = 0 \text{ for some } j \in I,\\ x^{(i)} \wedge y^{(i)}\,, & \text{otherwise,} \end{cases}
\end{aligned} \tag{4.1}$$

for each $x, y \in L$ and each $i \in I$.

Proof. For an element x of L, abbreviate by "$x_{(i)}$ exists" (resp., "$x^{(i)}$ exists") the statement that $L_i \downarrow x$ is a principal ideal in L_i (resp., $\overline{L}_i \uparrow x$ is a principal filter in $\overline{L}_i$), and then denote by $x_{(i)}$ (resp., $x^{(i)}$) the largest element of $L_i \downarrow x$ (resp., the least element of $\overline{L}_i \uparrow x$). Denote by K the set of all $x \in L$ such that both $x_{(i)}$ and $x^{(i)}$ exist for each $i \in I$. It is clear that K contains P and that both $p_{(i)}$ and $p^{(i)}$ are given by the first two formulas of (4.1), for any $p \in P$. Furthermore, it follows immediately from the definition of K that

$$\mathcal{I}(z) = \bigcup_{i\in I} (L_i \downarrow z_{(i)}), \tag{4.2}$$

$$\mathcal{F}(z) = \bigcup_{i\in I} (L_i \uparrow z^{(i)}), \tag{4.3}$$

for each $z \in K$. We shall establish that K is a sublattice of L. So let $x, y \in K$, put $u = x \wedge y$ and $v = x \vee y$. It is straightforward that for each $i \in I$, both $u_{(i)}$ and $v^{(i)}$ exist, and

$$u_{(i)} = x_{(i)} \wedge y_{(i)}, \quad v^{(i)} = x^{(i)} \vee y^{(i)}. \tag{4.4}$$

Now we shall prove that $v_{(i)}$ exists and is equal to $x_{(i)} \vee y_{(i)}$. By the induction hypothesis, (4.2) holds at both x and y. So, as $\mathcal{I}(v) = \mathcal{I}(x) \vee \mathcal{I}(y)$, in order to get the asserted existence and description of the elements $v_{(i)}$, it suffices to prove that

$$\bigcup_{i\in I} (L_i \downarrow x_{(i)}) \vee \bigcup_{i\in I} (L_i \downarrow y_{(i)}) = \bigcup_{i\in I} \big(L_i \downarrow (x_{(i)} \vee y_{(i)})\big). \tag{4.5}$$

The containment from left to right is obvious, and each $x_{(i)} \vee y_{(i)}$ is contained in any o-ideal of P containing $\{x_{(i)}, y_{(i)}\}$, so it suffices to prove that the right hand side of (4.5) is an o-ideal of P. As the join operation in P is internal to each L_i, this set is closed under joins. As each L_i is a lower subset of P, this set is also a lower subset of P. This establishes the desired result for the $v_{(i)}$s.

It remains to prove that $u^{(i)}$ exists and is equal to z_i, where $z_i = x^{(i)} \wedge y^{(i)}$ if $x^{(j)} \wedge y^{(j)} \neq 0$ for all j, and $z_i = 0$ otherwise. By the induction hypothesis, (4.3) holds at both x and y. So, as $\mathcal{F}(u) = \mathcal{F}(x) \vee \mathcal{F}(y)$, in order to get the asserted existence and description of the elements $u^{(i)}$, it suffices to prove that

$$\bigcup_{i\in I}(L_i \uparrow x^{(i)}) \vee \bigcup_{i\in I}(L_i \uparrow y^{(i)}) = \bigcup_{i\in I}(L_i \uparrow z_i). \tag{4.6}$$

The containment from left to right is obvious. If an o-filter U of P contains $\{x^{(i)}, y^{(i)}\}$ for all $i \in I$, then it also contains all elements $x^{(i)} \wedge y^{(i)}$; in particular, it is equal to P in case $x^{(i)} \wedge y^{(i)} = 0$ for some i. In any case, $z_i \in U$ for all $i \in I$. So it suffices to prove that the right hand side of (4.6) is an o-filter of P. This is trivial in case $z_i = 0$ for some i, so suppose that $z_i \neq 0$ for all i. As the meet operation in P is internal to each L_i, the right hand side of (4.6) is closed under meets. As each $L_i \setminus \{0\}$ is an upper subset of P, this set is also an upper subset of P. This establishes the desired result for the $u^{(i)}$s. □

Lemma 4.2. *Let K_i be a 0-sublattice of a lattice L_i, for each $i \in I$. Then the canonical 0-lattice homomorphism $f\colon \coprod^0_{i\in I} K_i \to \coprod^0_{i\in I} L_i$ is an embedding.*

Proof. By the amalgamation property for lattices [7, Section V.4], the i-th coprojection from K_i to K is an embedding, for each $i \in I$. Put $L'_i = K \amalg_{K_i} L_i$ for each $i \in I$. Comparing the universal properties, it is immediate that the 0-coproduct L of $(L_i \mid i \in I)$ is also the coproduct of $(L'_i \mid i \in I)$ over K. Again by using the amalgamation property for lattices, all canonical maps from the L'_is to L are embeddings. So, in particular, the canonical map from their common sublattice K to L is an embedding. □

We shall call the adjoint maps $\alpha_i\colon x \mapsto x_{(i)}$ and $\beta_i\colon x \mapsto x^{(i)}$ the canonical lower, resp. upper adjoint of L onto L_i, resp. $\overline{L}_i$. Observe that these maps may not be defined in the case of amalgamation of two lattices over a common sublattice, as Example 5.3 will show. (In that example, there is no largest element of B below $b_0 \vee c_0$.)

The following result is an immediate consequence of well-known general properties of adjoint maps.

Corollary 4.3. *The canonical embedding from L_i into L is both lower bounded and upper bounded, for each $i \in I$. In particular, it is a nonempty-complete lattice homomorphism. Furthermore, the lower adjoint α_i is meet-complete while the upper adjoint β_i is nonempty-join-complete.*

In the following lemma, we shall represent the elements of $L = \coprod^0_{i\in I} L_i$ in the form $\mathbf{p}(\vec{a})$, where $\mathbf{p}$ is a lattice term with variables from $I \times \omega$ and the "vector" $\vec{a} = (a_{i,n} \mid (i,n) \in I \times \omega)$ is an element of the cartesian product $\Pi = \prod_{(i,n)\in I\times\omega} L_i$. Define a *support* of $\mathbf{p}$ as a subset J of I such that $\mathbf{p}$ involves only variables from $J\times\omega$. Obviously, $\mathbf{p}$ has a finite support. It is straightforward from (4.1) that $\mathbf{p}(\vec{a})_{(i)} = 0$ and either $\mathbf{p}(\vec{a}) = 0$ or $\mathbf{p}(\vec{a})^{(i)} = \infty$, for each i outside a support of $\mathbf{p}$.

Lemma 4.4. *Let Λ be an upward directed poset, let $(\vec{a}^\lambda \mid \lambda \in \Lambda)$ be an isotone family of elements of Π with supremum $\vec{a}$ in Π, and let $\mathbf{p}$ be a lattice term. If all the lattices L_i are upper continuous, then $\mathbf{p}(\vec{a}) = \bigvee_{\lambda\in\Lambda} \mathbf{p}(\vec{a}^\lambda)$ in L.*

Again, Example 5.3 will show that Lemma 4.4 fails to extend to the amalgam of two lattices over a common ideal.

Proof. As $\mathbf{p}(\vec{a})$ is clearly an upper bound for all elements $\mathbf{p}(\vec{a}^\lambda)$, it suffices to prove that for each lattice term $\mathbf{q}$ on $I\times\omega$ and each $\vec{b}\in\Pi$ such that $\mathbf{p}(\vec{a}^\lambda)\leq\mathbf{q}(\vec{b})$ for all $\lambda\in\Lambda$, the inequality $\mathbf{p}(\vec{a})\leq\mathbf{q}(\vec{b})$ holds. We argue by induction on the sums of the lengths of $\mathbf{p}$ and $\mathbf{q}$. The case where $\mathbf{p}$ is a projection follows immediately from the second sentence of Corollary 4.3. The case where either $\mathbf{p}$ is a join or $\mathbf{q}$ is a meet is straightforward.

Now suppose that $\mathbf{p}=\mathbf{p}_0\wedge\mathbf{p}_1$ and $\mathbf{q}=\mathbf{q}_0\vee\mathbf{q}_1$. We shall make repeated uses of the following easily established principle, which uses only the assumption that Λ is upward directed:

> *For every positive integer n and every $X_0,\dots,X_{n-1}\subseteq\Lambda$, if $\bigcup_{i<n}X_i$ is cofinal in Λ, then one of the X_is is cofinal in Λ.*

Now we use (3.3). If there exists a cofinal subset Λ' of Λ such that

$$(\forall\lambda\in\Lambda')(\exists i<2)\big(\text{either }\mathbf{p}_i(\vec{a}^\lambda)\leq\mathbf{q}(\vec{b})\text{ or }\mathbf{p}(\vec{a}^\lambda)\leq\mathbf{q}_i(\vec{b})\big),$$

then there are $i<2$ and a smaller cofinal subset Λ'' of Λ' such that

$$\begin{aligned}\text{either}\quad&(\forall\lambda\in\Lambda'')\big(\mathbf{p}_i(\vec{a}^\lambda)\leq\mathbf{q}(\vec{b})\big)\\ \text{or}\quad&(\forall\lambda\in\Lambda'')\big(\mathbf{p}(\vec{a}^\lambda)\leq\mathbf{q}_i(\vec{b})\big).\end{aligned}$$

In the first case, it follows from the induction hypothesis that $\mathbf{p}_i(\vec{a})\leq\mathbf{q}(\vec{b})$. In the second case, it follows from the induction hypothesis that $\mathbf{p}(\vec{a})\leq\mathbf{q}_i(\vec{b})$. In both cases, $\mathbf{p}(\vec{a})\leq\mathbf{q}(\vec{b})$. It remains to consider the case where there exists a cofinal subset Λ' of Λ such that

$$(\forall\lambda\in\Lambda')(\exists c_\lambda\in P)\big(\mathbf{p}(\vec{a}^\lambda)\leq c_\lambda\leq\mathbf{q}(\vec{b})\big).$$

It follows from the induction hypothesis that

$$\mathbf{p}_\ell(\vec{a})=\bigvee_{\lambda\in\Lambda'}\mathbf{p}_\ell(\vec{a}^\lambda)\,,\quad\text{for all }\ell<2\,.\tag{4.7}$$

Fix a common finite support J of $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{q}_0$, $\mathbf{q}_1$. Each c_λ belongs to L_i, for some i in the given support J. By using the finiteness of J and by extracting a further cofinal subset of Λ', we may assume that all those i are equal to the same index $j\in J$. Hence we have reduced the problem to the case where

$$(\forall\lambda\in\Lambda')\big(\mathbf{p}(\vec{a}^\lambda)\leq c_\lambda\leq\mathbf{q}(\vec{b})\big)\,,\quad\text{where }c_\lambda=\mathbf{p}(\vec{a}^\lambda)^{(j)}\in L_j.\tag{4.8}$$

If $\mathbf{p}_0(\vec{a})^{(i)}\wedge\mathbf{p}_1(\vec{a})^{(i)}=0$ for some $i\in I$, then $\mathbf{p}(\vec{a})=0\leq\mathbf{q}(\vec{b})$ and we are done. Now suppose that $\mathbf{p}_0(\vec{a})^{(i)}\wedge\mathbf{p}_1(\vec{a})^{(i)}\neq0$ for all $i\in I$. By using (4.7), the finiteness of J, and the upper continuity of L_i, we obtain that there exists a cofinal subset Λ'' of Λ' such that

$$(\forall\lambda\in\Lambda'')(\forall i\in J)\big(\mathbf{p}_0(\vec{a}^\lambda)^{(i)}\wedge\mathbf{p}_1(\vec{a}^\lambda)^{(i)}\neq0\big).$$

In particular, both $\mathbf{p}_0(\vec{a}^\lambda)$ and $\mathbf{p}_1(\vec{a}^\lambda)$ are nonzero for each $\lambda\in\Lambda''$. As J is a common support of $\mathbf{p}_0$ and $\mathbf{p}_1$, the equality $\mathbf{p}_0(\vec{a}^\lambda)^{(i)}\wedge\mathbf{p}_1(\vec{a}^\lambda)^{(i)}=\infty$ holds for all $\lambda\in\Lambda''$ and all $i\in I\setminus J$, hence

$$(\forall\lambda\in\Lambda'')(\forall i\in I)\big(\mathbf{p}_0(\vec{a}^\lambda)^{(i)}\wedge\mathbf{p}_1(\vec{a}^\lambda)^{(i)}\neq0\big).$$

Thus it follows from (4.1) that $c_\lambda = \mathbf{p}(\vec{a}^\lambda)^{(j)} = \mathbf{p}_0(\vec{a}^\lambda)^{(j)} \wedge \mathbf{p}_1(\vec{a}^\lambda)^{(j)}$ for each $\lambda \in \Lambda''$. Hence, by the upper continuity of L_j (and thus of $\overline{L}_j$), (4.7), and the previously observed fact that the upper adjoint β_j is nonempty-join-complete, $\{c_\lambda \mid \lambda \in \Lambda''\}$ has a join in L_j, which is equal to $\mathbf{p}_0(\vec{a})^{(j)} \wedge \mathbf{p}_1(\vec{a})^{(j)} = \mathbf{p}(\vec{a})^{(j)}$. Therefore, it follows from (4.8) that $\mathbf{p}(\vec{a}) \leq \mathbf{p}(\vec{a})^{(j)} \leq \mathbf{q}(\vec{b})$. □

5. Ideal lattices and 0-coproducts

In this section we fix again a family $(L_i \mid i \in I)$ of lattices with zero, pairwise intersecting in $\{0\}$, and we form $L = \coprod^0_{i\in I} L_i$. We denote by $\varepsilon_i \colon \operatorname{Id} L_i \hookrightarrow \operatorname{Id} L$ the 0-lattice homomorphism induced by the canonical embedding $L_i \hookrightarrow L$, for each $i \in I$. By the universal property of the coproduct, there exists a unique 0-lattice homomorphism $\varepsilon \colon \coprod^0_{i\in I} \operatorname{Id} L_i \to \operatorname{Id} L$ such that $\varepsilon_i = \varepsilon\restriction_{\operatorname{Id} L_i}$ for each $i \in I$. Observe that in case I is finite, the lattice $\coprod^0_{i\in I} \operatorname{Id} L_i$ has $\bigvee_{i\in I} L_i$ as a largest element, and this element is sent by ε to L (because every element of L lies below some join of elements of the L_is). Hence, *if the index set I is finite, then the map ε preserves the unit as well.*

Lemma 5.1. *Let $\mathbf{p}$ be a lattice term on $I \times \omega$ and let $\vec{X} = (X_{i,n} \mid (i,n) \in I \times \omega)$ be an element of $\prod_{(i,n)\in I\times\omega} \operatorname{Id} L_i$. We put $\vec{\varepsilon}\vec{X} = (\varepsilon_i(X_{i,n}) \mid (i,n) \in I\times\omega) \in (\operatorname{Id} L)^{I\times\omega}$. Then the following equality holds.*

$$\mathbf{p}(\vec{\varepsilon}\vec{X}) = L \downarrow \{\mathbf{p}(\vec{x}) \mid \vec{x} \mathrel{\vec{\in}} \vec{X}\}\,,$$

where "$\vec{x} \mathrel{\vec{\in}} \vec{X}$" stands for $(\forall (i,n) \in I \times \omega)(x_{i,n} \in X_{i,n})$.

Proof. We argue by induction on the length of the term $\mathbf{p}$. If $\mathbf{p}$ is a projection, then the result follows immediately from the definition of the maps ε_i. If $\mathbf{p}$ is either a join or a meet, then the result follows immediately from the expressions for the join and the meet in the ideal lattice of L, in a fashion similar to the end of the proof of [7, Lemma I.4.8]. □

Theorem 5.2. *The canonical map $\varepsilon \colon \coprod^0_{i\in I} \operatorname{Id} L_i \to \operatorname{Id}\bigl(\coprod^0_{i\in I} L_i\bigr)$ is a 0-lattice embedding.*

Proof. We put again $L = \coprod^0_{i\in I} L_i$. Let $\mathbf{p}$, $\mathbf{q}$ be lattice terms in $I \times \omega$ and let $\vec{X} \in \prod_{(i,n)\in I\times\omega} \operatorname{Id} L_i$ such that $\mathbf{p}(\vec{\varepsilon}\vec{X}) \leq \mathbf{q}(\vec{\varepsilon}\vec{X})$ in $\operatorname{Id} L$. We must prove that $\mathbf{p}(\vec{X}) \leq \mathbf{q}(\vec{X})$ in $\coprod^0_{i\in I} \operatorname{Id} L_i$. For each $\vec{x} \mathrel{\vec{\in}} \vec{X}$, the inequalities $L \downarrow \mathbf{p}(\vec{x}) \leq \mathbf{p}(\vec{\varepsilon}\vec{X}) \leq \mathbf{q}(\vec{\varepsilon}\vec{X})$ hold in $\operatorname{Id} L$, thus, by Lemma 5.1, there exists $\vec{y} \mathrel{\vec{\in}} \vec{X}$ such that $L \downarrow \mathbf{p}(\vec{x}) \leq L \downarrow \mathbf{q}(\vec{y})$ in $\operatorname{Id} L$, that is, $\mathbf{p}(\vec{x}) \leq \mathbf{q}(\vec{y})$ in L. Therefore, by applying the canonical map from $L = \coprod^0_{i\in I} L_i$ to $\coprod^0_{i\in I} \operatorname{Id} L_i$ and putting $\vec{L} \downarrow \vec{x} = (L_i \downarrow x_{i,n} \mid (i,n) \in I \times \omega)$, we obtain

$$\mathbf{p}(\vec{L} \downarrow \vec{x}) \leq \mathbf{q}(\vec{L} \downarrow \vec{y}) \leq \mathbf{q}(\vec{X}) \quad \text{in } \coprod^0_{i\in I} \operatorname{Id} L_i\,. \tag{5.1}$$

As $\vec{X}$ is equal to the directed join $\bigvee_{\vec{x} \vec{\in} \vec{X}} (\vec{L} \downarrow \vec{x})$ in $\prod_{(i,n)\in I\times\omega} \operatorname{Id} L_i$ and each $\operatorname{Id} L_i$ is upper continuous, it follows from Lemma 4.4 that

$$\mathbf{p}(\vec{X}) = \bigvee \Bigl(\mathbf{p}(\vec{L} \downarrow \vec{x}) \mid \vec{x} \mathrel{\vec{\in}} \vec{X}\Bigr) \quad \text{in } \coprod^0_{i\in I} \operatorname{Id} L_i\,.$$

Therefore, it follows from (5.1) that

$$\mathbf{p}(\vec{X}) \leq \mathbf{q}(\vec{X}) \quad \text{in } \coprod^0_{i\in I} \operatorname{Id} L_i\,.$$ □

The following example shows that Theorem 5.2 does not extend to the amalgam $B \amalg_A C$ of two lattices B and C above a common ideal A. The underlying idea can be traced back to Grätzer and Schmidt in [8, Section 5].

Example 5.3. *Lattices B and C with a common ideal A such that the canonical lattice homomorphism $f\colon (\operatorname{Id} B) \amalg_{\operatorname{Id} A} (\operatorname{Id} C) \to \operatorname{Id}\big(B \amalg_A C\big)$ is not one-to-one.*

Proof. Denote by K the poset represented in Figure 1. We claim that the subsets A, B, and C of K defined by

$$\begin{aligned} A &= \{a_n \mid n < \omega\} \cup \{p_n \mid n < \omega\} \cup \{q_n \mid n < \omega\}\,, \\ B &= A \cup \{b_n \mid n < \omega\}\,, \\ C &= A \cup \{c_n \mid n < \omega\}\,. \end{aligned}$$

are as required. Observe that B and C are isomorphic lattices and that A is an ideal of both B and C.

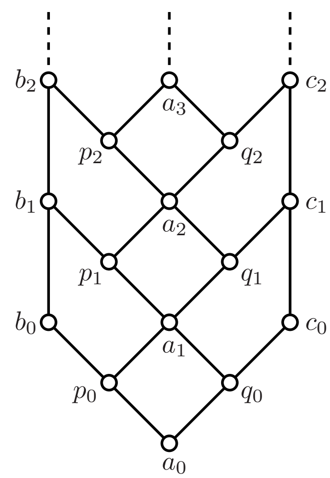

FIGURE 1. The poset K.

The map f is the unique lattice homomorphism that makes the diagram of Figure 2 commute. Unlabeled arrows are the corresponding canonical maps.

$$\begin{tikzcd}
 & \operatorname{Id} B \arrow[dr] \arrow[drr] & & \\
\operatorname{Id} A \arrow[ur] \arrow[dr] & & (\operatorname{Id} B) \amalg_{\operatorname{Id} A} (\operatorname{Id} C) \arrow[r, "f"] & \operatorname{Id}\big(B \amalg_A C\big) \\
 & \operatorname{Id} C \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

FIGURE 2. The commutative diagram defining the homomorphism f.

Put $D = B \amalg_A C$ and identify B and C with their images in D. Further, we endow $\operatorname{Id} B \cup \operatorname{Id} C$ with its natural structure of partial lattice, that is, the ordering is

the union of the orderings of $\operatorname{Id} B$ and $\operatorname{Id} C$ (remember that $A = B \cap C$ is an ideal of both B and C) and the joins and meets are those taking place in either $\operatorname{Id} B$ or $\operatorname{Id} C$. Observe that $\operatorname{Id} A = \operatorname{Id} B \cap \operatorname{Id} C$ and $(\operatorname{Id} B) \amalg_{\operatorname{Id} A} (\operatorname{Id} C)$ is the free lattice on the partial lattice $(\operatorname{Id} B) \cup (\operatorname{Id} C)$. As the latter is identified with its canonical image in $(\operatorname{Id} B) \amalg_{\operatorname{Id} A} (\operatorname{Id} C)$, the elements A, $B \downarrow b_0$, and $C \downarrow c_0$ belong to $(\operatorname{Id} B) \amalg_{\operatorname{Id} A} (\operatorname{Id} C)$.

We prove by induction that $a_n \leq b_0 \vee c_0$ in D for all $n < \omega$. This is trivial for $n = 0$. Suppose that $a_n \leq b_0 \vee c_0$. Then $a_n \vee b_0 \leq b_0 \vee c_0$, but B is a sublattice of D containing the subset $\{a_n, b_0\}$ with join b_n, thus $b_n \leq b_0 \vee c_0$, and thus $p_n \leq b_0 \vee c_0$. Similarly, $q_n \leq b_0 \vee c_0$, but A is a sublattice of D containing the subset $\{p_n, q_n\}$ with join a_{n+1}, and thus $a_{n+1} \leq b_0 \vee c_0$, which completes the induction step.

So we have established the inequality

$$f(A) \leq f(B \downarrow b_0) \vee f(C \downarrow c_0) \quad \text{in } \operatorname{Id}\bigl(B \amalg_A C\bigr) = \operatorname{Id} D\,. \tag{5.2}$$

Now observe that $\{B \downarrow x \mid x \in B\} \cup \{C \downarrow y \mid y \in C\}$ is an o-ideal of the partial lattice $(\operatorname{Id} B) \cup (\operatorname{Id} C)$, containing $\{B \downarrow b_0, C \downarrow c_0\}$ and to which A does not belong. Hence, $A \notin \mathfrak{I}(\{B \downarrow b_0, C \downarrow c_0\})$, which means that $A \nleq (B \downarrow b_0) \vee (C \downarrow c_0)$ in $(\operatorname{Id} B) \amalg_{\operatorname{Id} A} (\operatorname{Id} C)$. Therefore, by (5.2), f is not an embedding. $\square$

As observed before, this example shows that Lemma 4.4 fails to extend to the amalgam of two lattices over a common ideal. Indeed, while $A = \bigvee_n (A \downarrow a_n)$ in $\operatorname{Id} B$, the same equality fails in $(\operatorname{Id} B) \amalg_{\operatorname{Id} A} (\operatorname{Id} C)$. The reason for this is that $A \downarrow a_n \leq (B \downarrow b_0) \vee (C \downarrow c_0)$ for each n, while $A \nleq (B \downarrow b_0) \vee (C \downarrow c_0)$.

6. Embedding coproducts of infinite partition lattices

Whitman's Embedding Theorem states that every lattice embeds into $\operatorname{Eq} \Omega$, for some set Ω. We shall use a proof of Whitman's Theorem due to B. Jónsson [9], see also [7, Section IV.4]. The following result is proved there.

Lemma 6.1. *For every lattice L with zero, there are an infinite set Ω and a map $\delta \colon \Omega \times \Omega \to L$ satisfying the following properties:*

(1) $\delta(x, y) = 0$ *iff* $x = y$, *for all* $x, y \in \Omega$.
(2) $\delta(x, y) = \delta(y, x)$, *for all* $x, y \in \Omega$.
(3) $\delta(x, z) \leq \delta(x, y) \vee \delta(y, z)$, *for all* $x, y, z \in L$.
(4) *For all* $x, y \in \Omega$ *and all* $a, b \in L$ *such that* $\delta(x, y) \leq a \vee b$, *there are* $z_1, z_2, z_3 \in \Omega$ *such that* $\delta(x, z_1) = a$, $\delta(z_1, z_2) = b$, $\delta(z_2, z_3) = a$, *and* $\delta(z_3, y) = b$.

Observe, in particular, that the map δ is *surjective*. Furthermore, a straightforward Löwenheim-Skolem type argument ("keeping only the necessary elements in Ω") shows that one may take $\operatorname{card} \Omega = \operatorname{card} L + \aleph_0$.

The following is the basis for Jónsson's proof of Whitman's Embedding Theorem.

Corollary 6.2. *For every lattice L with zero and every set Ω such that $\operatorname{card} \Omega = \operatorname{card} L + \aleph_0$, there exists a complete lattice embedding from $\operatorname{Id} L$ into $\operatorname{Eq} \Omega$.*

Proof. Any map δ as in Lemma 6.1 gives rise to a map $\varphi \colon \operatorname{Id} L \to \operatorname{Eq} \Omega$ defined by the rule

$$\varphi(A) = \{(x, y) \in \Omega \times \Omega \mid \delta(x, y) \in A\}\,, \quad \text{for each } A \in \operatorname{Id} L\,, \tag{6.1}$$

and conditions (1)–(4) above imply that φ is a complete lattice embedding. $\square$

Theorem 6.3. *Let Ω be an infinite set. Then there exists a* $0,1$*-lattice embedding from* $(\operatorname{Eq}\Omega) \amalg^0 (\operatorname{Eq}\Omega)$ *into* $\operatorname{Eq}\Omega$*.*

Proof. Denote by K the sublattice of $\operatorname{Eq}\Omega$ consisting of all *compact* equivalence relations of Ω. Thus the elements of K are exactly the equivalence relations containing only finitely many non-diagonal pairs. In particular, $\operatorname{Eq}\Omega$ is canonically isomorphic to $\operatorname{Id} K$.

Now we apply Corollary 6.2 to $L = K \amalg^0 K$. As $\operatorname{card} L = \operatorname{card}\Omega$, we obtain a complete lattice embedding $\varphi\colon \operatorname{Id} L \hookrightarrow \operatorname{Eq}\Omega$. However, $\operatorname{Id} L = \operatorname{Id}(K \amalg^0 K)$ contains, by Theorem 5.2 and the last sentence of the first paragraph of Section 5, a $0,1$-sublattice isomorphic to $(\operatorname{Id} K) \amalg^0 (\operatorname{Id} K)$, thus to $(\operatorname{Eq}\Omega) \amalg^0 (\operatorname{Eq}\Omega)$. $\square$

For any nonempty set Ω, form $\overline{\Omega} = \Omega \cup \{\infty\}$ for an outside point ∞. As there exists a retraction $\rho\colon \overline{\Omega} \twoheadrightarrow \Omega$ (pick $p \in \Omega$ and send ∞ to p), we can form a meet-complete, nonempty-join-complete lattice embedding $\eta\colon \operatorname{Eq}\Omega \hookrightarrow \operatorname{Eq}\overline{\Omega}$ by setting

$$\eta(\theta) = \{(x,y) \in \overline{\Omega} \times \overline{\Omega} \mid (\rho(x),\rho(y)) \in \theta\}\,, \quad \text{for each } \theta \in \operatorname{Eq}\Omega\,,$$

and η sends the zero element of $\operatorname{Eq}\Omega$ to a nonzero element of $\operatorname{Eq}\overline{\Omega}$. Hence, in case Ω is infinite, $(\operatorname{Eq}\Omega)^\circ$ completely embeds into $\operatorname{Eq}\Omega$. As $(\operatorname{Eq}\Omega) \amalg (\operatorname{Eq}\Omega)$ is the sublattice of $(\operatorname{Eq}\Omega)^\circ \amalg^0 (\operatorname{Eq}\Omega)^\circ$ generated by the union of the images of $\operatorname{Eq}\Omega$ under the two canonical coprojections, it follows from Theorem 6.3 and Lemma 4.2 that $(\operatorname{Eq}\Omega) \amalg (\operatorname{Eq}\Omega)$ has a 1-lattice embedding into $\operatorname{Eq}\Omega$. If we denote by θ the image of zero under this embedding, then $(\operatorname{Eq}\Omega) \amalg (\operatorname{Eq}\Omega)$ has a $0,1$-lattice embedding into $\operatorname{Eq}(\Omega/\theta)$, and thus, as $\operatorname{card}(\Omega/\theta) \leq \operatorname{card}\Omega$, into $\operatorname{Eq}\Omega$. Hence we obtain

Theorem 6.4. *Let Ω be an infinite set. Then there exists a* $0,1$*-lattice embedding from* $(\operatorname{Eq}\Omega) \amalg (\operatorname{Eq}\Omega)$ *into* $\operatorname{Eq}\Omega$*.*

By applying the category equivalence $L \mapsto L^{\mathrm{op}}$ to Theorems 6.3 and 6.4 and denoting by $\amalg^1$ the coproduct of 1-lattices, we obtain the following result.

Theorem 6.5. *Let Ω be an infinite set. Then there are* $0,1$*-lattice embeddings from* $(\operatorname{Eq}\Omega)^{\mathrm{op}} \amalg^1 (\operatorname{Eq}\Omega)^{\mathrm{op}}$ *into* $(\operatorname{Eq}\Omega)^{\mathrm{op}}$ *and from* $(\operatorname{Eq}\Omega)^{\mathrm{op}} \amalg (\operatorname{Eq}\Omega)^{\mathrm{op}}$ *into* $(\operatorname{Eq}\Omega)^{\mathrm{op}}$*.*

By using the results of [2], we can now fit the copower of the optimal number of copies of $L = \operatorname{Eq}\Omega$ into itself. The variety $\mathbf{V}$ to which we apply those results is, of course, the variety of all lattices with zero. The functor to be considered sends every set I to $F(I) = \coprod_I^0 L$, the 0-coproduct of I copies of L. If we denote by $e_i^I\colon L \hookrightarrow F(I)$ the i-th coprojection, then, for any map $f\colon I \to J$, $F(f)$ is the unique 0-lattice homomorphism from $F(I)$ to $F(J)$ such that $F(f) \circ e_i^I = e_{f(i)}^J$ for all $i \in I$. Observe that even in case both I and J are finite, $F(f)$ does not preserve the unit unless f is surjective. The condition labeled (9) in [2, Section 3], stating that every element of $F(I)$ belongs to the range of $F(a)$ for some $a\colon n \to I$, for some positive integer n, is obviously satisfied. Hence, by [2, Theorem 3.1], $F(\mathfrak{P}(\Omega))$ has a 0-lattice embedding into $F(\omega)^\Omega$. Furthermore, it follows from [2, Lemma 3.3] that $F(\omega)$ has a 0-lattice embedding into $\prod_{1 \leq n < \omega} F(n)$. By Lemma 4.2 and Theorem 6.3, each $F(n)$ has a $0,1$-lattice embedding into L. As, by the final paragraph of [2, Section 2], L^Ω has a 0-lattice embedding into L, we obtain the following theorem.

Theorem 6.6. *Let Ω be an infinite set. Then the following statements hold:*

(1) $\coprod_{\mathfrak{P}(\Omega)}^0 \operatorname{Eq}\Omega$ *has a* 0*-lattice embedding into* $\operatorname{Eq}\Omega$*.*

(2) $\coprod_{\mathfrak{P}(\Omega)} \operatorname{Eq}\Omega$ *has a lattice embedding into* $\operatorname{Eq}\Omega$.
(3) $\coprod^1_{\mathfrak{P}(\Omega)} (\operatorname{Eq}\Omega)^{\mathrm{op}}$ *has a* 1*-lattice embedding into* $(\operatorname{Eq}\Omega)^{\mathrm{op}}$.
(4) $\coprod_{\mathfrak{P}(\Omega)} (\operatorname{Eq}\Omega)^{\mathrm{op}}$ *has a lattice embedding into* $(\operatorname{Eq}\Omega)^{\mathrm{op}}$.

This raises the question whether $(\operatorname{Eq}\Omega) \amalg^1 (\operatorname{Eq}\Omega)$ embeds into $\operatorname{Eq}\Omega$, which the methods of the present paper do not seem to settle in any obvious way. More generally, we do not know whether, for a sublattice A of $\operatorname{Eq}\Omega$, the amalgam $(\operatorname{Eq}\Omega) \amalg_A (\operatorname{Eq}\Omega)$ of two copies of $\operatorname{Eq}\Omega$ over A embeds into $\operatorname{Eq}\Omega$.

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